Belief CSP : a New CSP Framework under Uncertainty

Theoretical Foundations

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Abstract. The Constraint Satisfaction Problem (CSP) is acknowledged as a simple declarative formalism for modeling and solving well-defined decision problems. Still, the standard CSP has proven unsuited for reasoning under imperfection where flexibility is a key notion. With the aim of building a reliable model and by merging the belief function theory and the CSP formalism, in this paper, we introduce a new CSP extension, labeled Belief Constraint Satisfaction Problem (BCSP), fulfilling two tasks that are managing imperfection and modeling flexibility. The first task conserves the real core of the problem being tackled; the second one allows maintaining that core as the problem's environment is becoming more or less imperfect.

Keywords: CSP, Belief function theory, Belief CSP, Imperfection, Flexibility

1 Introduction

The CSP framework has carried high attention within the AI community because of its simplicity and generality. However, decision problems tackled by the classical CSP are assumed to be well-defined so that all their items are precise and known with certainty. Hence, the CSP has proven unfit for reasoning under imperfection[3] where any item may be uncertain and/or imprecise. The imperfect items may be deleted, ignored or reformed. Anyhow, we may find ourselves taking on the wrong problem and, thus, yielding irrelevant decisions. Therefore, in order to get a reliable model of the problem being tackled, we have to model that imperfection. Various CSP extensions have been made to deal with imperfection by exploiting uncertainty theories[3, 9, 10]. These proposals handle only one imperfection aspect, i.e., uncertainty or imprecision, and differently of what

[3] Uncertainty has been, mostly, used in the literature to refer to imperfection, whereas, imperfection implies both uncertainty which is an epistemic property of the relation between the information and the agent knowledge about the world, and imprecision that touches the information itself[14].

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we propose these latter, in the best cases, use two formalisms to express both imperfection and flexibility as do the Fuzzy CSP[2] which combines fuzzy sets and possibility theories under a commensurability assumption. However, there is no attempt to take advantage of the belief function theory[1, 11, 13] which offers a sound mathematical basis that manages simultaneously both imperfection aspects and enables both imperfection and flexibility to be expressed with the same formalism. In addition, the belief function theory allows an explicit management of both partial and total ignorance using only the available knowledge and no more. This paper is devoted to introduce the theoretical foundations of a new CSP extension labeled BCSP reaping benefits from both CSP and belief function theory and fulfilling two tasks that are managing imperfect (uncertain and/or imprecise) relations and modeling soft or flexible constraints.
The paper is organized as follows: in the section 2 we, succinctly, recall the formal definition of the CSP besides some related notions. Then, we overview the basics of the belief function theory as a tool for modeling imperfection and flexibility within the CSP. In the section 3, we introduce the BCSP, where, the main concepts are illustrated by a simple example.
In this paper, we want to focus on the theoretical foundations of the BCSP, where comparison with other CSP extensions and experimental results will be introduced in a forthcoming paper.

2 Background Concepts

2.1 Constraint Satisfaction Problem (CSP)

A classical CSP is defined by a quadruplet (X, D, C, R) where $X = \{x_1, ..., x_n\}$ is a finite set of n variables, each x_i takes its values in a finite domain D_i such that $D = \{D_1, ..., D_n\}$. The simultaneous assignment of values to a set of variables is called an instantiation and denoted by θ. $C = \{C_1, ..., C_m\}$ is a finite set of m constraints where each constraint C_i is defined on a subset of variables $S_i \subseteq X$ delimited its scope and by a relation R_i that specifies the set of compatible instantiations with C_i; R_i is a subset of the cartesian product of the domains of the variables in S_i (i.e., $R_i \subseteq \times\{D_i \mid x_i \subseteq S_i\}$)[4]. A constraint C_i is said to be satisfied by an instantiation θ defined on a set of variables V iff $S_i \subseteq V$ and $\exists\theta_i \in R_i$ such that $\theta_i \subseteq \theta$. The main task is to find a solution, that is an instantiation of all variables so that all constraints are satisfied. We denote the set of all solutions of a given CSP P by $Sols(P)$. A CSP is said to be consistent iff it has at least one solution, otherwise, it is said to be inconsistent.
It is important to note that, besides handling imperfection, our proposal is based on the flexibility notion in CSPs, more precisely, in the constraints. Hence, we should distinguish between hard and soft constraints. Classically, a constraint is a yes-or-no matter where it enumerates the certainly compatible instantiations. Thus, hard constraint is considered as imperative so that every solution should satisfy, whereas, soft or flexible constraint can be satisfied to some degree.

2.2 Belief Function Theory

The belief function theory was first initiated by Dempster[1] and then extended by Shafer[11]. Several interpretations have been introduced[18, 7]. The Transferable Belief Model(TBM) is a non-probabilistic interpretation established by Smets[15] that allows a clear separation between knowledge modeling and decision making since the reasoning process is illustrated through two levels: the credal level where the agent knowledge is represented in the static part and manipulated in the dynamic part; and the pignistic level where the decision part is considered aside. This two-level reasoning enables us to, explicitly, handle imperfection and flexibility within the CSP, all with the same formalism.
Likewise, it is important to distinguish the univariate formalism that handles one variable and the multivariate one[8] where the problem is modeled via multiple variables which, analogically, coincides with the CSP structcture.

Static part. Let $X = \{x_1, ..., x_n\}$ be a finite set of variables, where, each variable x_i is associated to a set of mutually exclusive but not necessarily exhaustive realizations, called frame of discernment[4], denoted by Θ_i. Given a non-empty subset S of X, its frame of discernment, denoted by Θ_S is obtained by the cartesian product of the frames of discernment of the involved variables (i.e., $\Theta_S = \times\{\Theta_i \in S\}$). The elements of Θ_S are called configurations of S on which we induce a valuation function called basic belief assignment (bba) that represents some knowledge (complete, partial or ignorant) about the variables in S. the bba m divides belief over singletons and subsets, i.e., the power set, of Θ_S as follows:

$$m : 2^{\Theta_S} \longrightarrow [0, 1] \quad \text{such that} \sum_{C \subseteq \Theta} m(C) = 1 \tag{1}$$

The value $m(C)$ called basic belief mass(bbm) represents the part of belief, exactly, committed to C[11]. Every configuration subset whose bbm is strictly positive is called focal element.

Dynamic Part. Two basic operations may be performed on bba function, that are, the vacuous extension which corresponds to knowledge refinement and the marginalization which coincides with knowledge coarsening[11].
The vacuous extension of a bba m defined on S, to a larger set $S^{'}$, such that $S \subseteq S^{'}$ induces a bba $m^{S \uparrow S^{'}}$ defined on $S^{'}$ as follows:

$$m^{S \uparrow S^{'}}(B) = \begin{cases} m(A) \text{ if } B = A^{\uparrow S^{'}} \\ 0 \text{ otherwise} \end{cases} \quad \text{for all } A \subseteq \Theta_S. \tag{2}$$

such that $A^{\uparrow S'} = A \times \Theta_{S'-S}$.
The marginalization is a projection of a bba $m^{'}$ defined on $S^{'}$ into a bba $m^{S^{'} \downarrow S}$ defined on a coarser set S, i.e.,$S \subseteq S^{'}$.

$$m^{S^{'} \downarrow S}(B) = \sum_{A \subseteq \Theta_{S'} : A^{\downarrow S} = B} m^{'}(A) \text{ for all B } \subseteq \Theta_S \tag{3}$$

[4] The frame of discernment Θ_i corresponds, by analogy, to the domain D_i of the variable x_i in a CSP.

such that $A^{\downarrow S}$ is obtained by removing all realizations of each configuration of A which corresponds to variables in $(S' - S)$.

Decision making. As states the decision theory[6], as well as the Dutch book arguments (DBA)[5], a rationally consistent decision must be casted using probabilities. Thus, the credal bbm is reformed to a probability measure known as the pignistic probability using the pignistic transformation[13, 16]. Let m be a bbm defined on Θ, the produced pignistic probability, denoted by $BetP$, is defined as follows:

$$BetP(A) = \sum_{B \subseteq \Theta} \left(m(B) \frac{|A \cap B}{|B|(1 - m(\emptyset))} \right) \text{,for all } A \in \Theta. \qquad (4)$$

where $|A|$ denotes the number of elements of Θ in A.

3 Belief Constraint Satisfaction Problem (BCSP)

3.1 Definition

A Belief Constraint Satisfaction Problem (BCSP) is a quadruplet (X, D, C, R), where, $X = \{x_1, ..., x_n\}$ is a finite set of variables; $D = \{D_1, ..., D_n\}$ is the set of their domains, such that, D_i is the domain associated to the variable x_i; $C = \{C_1, ..., C_m\}$ is a finite set of belief constraints, each constraint C_i is defined by the pair (S_i, R_i), such that, $S_i \subseteq X$ represents its scope and R_i is its imperfect relation, that is an uncertain and/or imprecise relation.
As we have adopted the TBM, in the credal level imperfect relations are expressed as the static part; some useful operations on imperfect relations are represented in the dynamic part; then, in the pignistic level the belief constraints are induced and solutions are computed.

3.2 Imperfect relation

A belief constraint C_i is defined by the pair (S_i, R_i), where, S_i is a subset of the problem variables delimiting the belief constraint scope, i.e., $S_i = \{X_{i_1}, ..., X_{i_k}\} \subseteq X$, and R_i which is, in turn, defined by the pair (V_i, Θ_i) represents an imperfect (uncertain and/or imprecise) relation that associates a valuation V_i, the bba function, over the frame of discernment of the belief constraint Θ_i obtained by the cartesian product of the involved variables domains, i.e., $\Theta_i = D_{i_1} \times ... \times D_{i_k}$. The valuation V_i is defined as follows:

$$V_i = m_i : 2^{\Theta_i} \rightarrow [0, 1] \mid \sum_{I \subseteq \Theta_i} m_i(I) = 1 \qquad (5)$$

where I is a singleton or a subset of instantiations.
If we define an instantiation as a logical relation between variables, the finite amount of support enclosed in the bba (i.e., $m_i(I)$) and derived from the available pieces of evidence can be interpreted as the potentiality degree of a given relation to, actually, occur so that the potentiality of the belief constraint to

be satisfied by such a variables instantiation(s), I. A second reading interprets this bbm distribution as preference levels induced over instantiations. Formally, let θ_1 and θ_2 two subsets of Θ_i (i.e., $\theta_1, \theta_2 \subseteq \Theta_i$), $m(\theta_1) > m(\theta_2)$ means that θ_1 is more believable (certain) than θ_2 and hence θ_1 is, a priori, preferred to θ_2 for the satisfaction of the belief constraint. This latter reading shows that the bba function has twofold role. The first is to quantify our belief about a given instantiation whereas the second is to induce a preorder among them.
According to the "closed world assumption" established by Shafer[11], an imperfect relation R_i is said to be normalized iff $m_i(\emptyset) = 0$. Otherwise, it is said to be unnormalized. This assumption is later relaxed by Smets[15] as the "open world assumption" where the empty set bbm quantifies the conflict amount between the beliefs on Θ_i.

An example. In order to illustrate the different notions related to the BCSP, let us tackle the problem "Pacifist or not" adapted from[12]. We would like to determine whether a person is a pacifist. The available knowledge indicates that most Republicans are not pacifists and we are 90 percent certain about this. Moreover, we are 99 percent sure that most Quakers are pacifists. The corresponding BCSP of this problem would be the quadruple (X, D, C, R) where:

- $X = \{Pa, Re, Qu\}$ the set of variables, such that, we use Pa to denote Pacifist, Re for Republican and Qu for Quaker;
- $D = \{D_{Pa}, D_{Re}, D_{Qu}\}$, such that, $D_{Pa} = \{p(Pacifist), \bar{p}(notPacifist)\}$; $D_{Re} = \{r(Republican), \bar{r}(notRepublican)\}$ and $D_{Qu} = \{q(Quaker), \bar{q}(notQuaker)\}$ which define, respectively, the domains of the variables Pa, Qu and Re ;
- $C = \{C_1, C_2\}$ the set of belief constraints;
- $R = \{R_1, R_2\}$, the set of imperfect relations, such that, $S_1 = \{Re, Pa\}$ and $S_2 = \{Qu, Pa\}$; $\Theta_1 = \{(r, p), (r, \bar{p}), (\bar{r}, p), (\bar{r}, \bar{p})\}$ and $\Theta_2 = (q, p), (q, \bar{p}), (\bar{q}, p), (\bar{q}, \bar{p})$ R_1(See Table 1), R_2(See Table 2)

Table 1. The bba function for R_1

2^{Θ_1}	m_1
$\{(r, \bar{p}), (\bar{r}, p), (\bar{r}, \bar{p})\}$	0.9
$\{(r, p), (r, \bar{p}), (\bar{r}, p), (\bar{r}, \bar{p})\}$	0.1

As we can, obviously, note both of the relations R_1 and R_2 are normalized.

Special relations. The most appealing feature that makes the belief function theory an efficient tool is its faithfulness in recognition our knowledge as well as our ignorance. That is why both of extreme knowledge cases, namely, the total knowledge and the total ignorance within the notion of belief constraints are easily expressed.

Table 2. The bba function for R_2

2^{Θ_2}	m_2
$\{(q,p),(\bar{q},p),(\bar{q},\bar{p})\}$	0.99
$\{(q,p),(\bar{q},p),(q,\bar{p}),(\bar{q},\bar{p})\}$	0.01

- **Complete certainty (perfect relation : certain and precise)** When the sought for instantiation is perfectly known and unique with reference to a given belief constraint, the associated relation is represented using the certain belief function where the one and only focal element is that instantiation. Consider a belief constraint C_i, which is described by the certain and precise relation R_i, the associated bba m_i is defined as follows:
  $$\exists\theta \subset \Theta_i, |\theta| = 1 \text{ such that } m_i(\theta) = 1 \text{ and } m_i(\varphi) = 0, \forall\varphi \subseteq \Theta_i, \varphi \neq \theta \quad (6)$$
  Obviously, such a case of total knowledge fits the classical notion of perfect relation where all tuples are known with certainty. Hence, our belief model permits, as well, the formalizing of the standard CSP. For instance,
  $$\begin{cases} m\{(\bar{r},p)\} = 1 \\ m\{\varphi\} = 0, \forall\varphi \subseteq \Theta_1, \varphi \neq (\bar{r},p) \end{cases}$$
  is a certain and precise relation that may describe the belief constraint C_1 of our example.
- **Complete ignorance (imperfect relation: uncertain and imprecise)** When we have no information on what the instantiation(s) satisfying a given belief constraint could be, in other words, we are not able to establish any valuation on the variables tuples, we have recourse to the vacuous belief function. If C_i is a belief constraint, R_i is its associated relation which is described by the vacuous bba m_i that is defined as follows:
  $$m_i(\Theta_i) = 1 \text{ and } m_i(\theta) = 0, \forall\theta \neq \Theta_i \quad (7)$$
  For example,
  $$\begin{cases} m\{(r,p),(r,\bar{p}),(\bar{r},p),(\bar{r},\bar{p})\} = 1 \\ m(\theta) = 0, \forall\theta \neq \Theta_1 \end{cases}$$
  is a vacuous relation that may describe the belief constraint C_1 of our example. We are indifferent toward all the tuples, so that, all are believable.
- **Partial ignorance (imperfect relation : certain and imprecise)** The intermediate case between those two extreme cases is the partial ignorance. Such a case is described using the categorical belief function. If C_i is a belief constraint, R_i is its associated relation which is described by the categorical bba m_i that is defined as follows:
  $$\exists\theta \subset \Theta_i, |\theta| > 1 \text{ such that } m_i(\theta) = 1 \text{ and } m_i(\varphi) = 0, \forall\varphi \subseteq \Theta_i, \varphi \neq \theta \quad (8)$$
  The following bba may be a categorical relation that describes the belief constraint C_1.
  $$\begin{cases} m\{(r,\bar{p}),(\bar{r},p),(\bar{r},\bar{p})\} = 1 \\ m(\varphi) = 0, \forall\varphi \neq (r,\bar{p}),(\bar{r},p),(\bar{r},\bar{p}) \end{cases}$$

Another case which may be described by the belief functions is the case when we have some precise but uncertain information. As far as we know, there is no special belief function that expresses this case but it is still always feasible. With such knowledge, more than one focal element is believable. If we retake the belief constraint C_1, a precise and uncertain relation may be as follows:

$$\begin{cases} m\{(r, \bar{p})\} = 0.8 \\ m(r, p) = 0.2 \end{cases}$$

Operations on imperfect relations.

- **Vacuous extension** The vacuous extension of an imperfect relation R_i defined on S_i, to a larger set $S_i^{'}$, such that $S_i \subseteq S_i^{'}$, is an imperfect relation $R_i^{(\uparrow S_i^{'})}$ defined on $S_i^{'}$ and obtained as follows:

$$m_i^{(\uparrow S_i^{'})}(\varphi) = \begin{cases} m_i(\theta) \text{ if } \varphi = \theta^{\uparrow S_i^{'}} \\ 0 \text{ otherwise} \end{cases} \text{for all } \theta \subseteq \Theta_i \quad (9)$$

Such that $\theta^{\uparrow S_i^{'}}$ denotes the cylindrical extension of the set θ to $S_i^{'}$. The vacuous extension is useful when we want to know to what extent a given instantiation may satisfy the belief constraint C_i. In fact, it corresponds to a refinement of knowledge. For a further explanation, let us take the imperfect relation R_1 describing the belief constraint C_1 defined on $S_1 = \{Re, Pa\}$ (see Table 1), its extension to $S_1^{'} = \{Re, Qu, Pa\}$ gives the imperfect relation $R_1^{(\uparrow S_1^{'})}$ as follows (see Table 3):

Table 3. The extended bba function for $R_1^{(\uparrow S_1^{'})}$

$2^{\Theta_1^{'}}$	$m_1^{(\uparrow S_1^{'})}$
$\{(r, \bar{p}, q), (\bar{r}, p, q), (\bar{r}, \bar{p}, q), (r, \bar{p}, \bar{q}), (\bar{r}, p, \bar{q}), (\bar{r}, \bar{p}, \bar{q})\}$	0.9
$\{(r, p, q), (r, \bar{p}, q), (\bar{r}, p, q), (\bar{r}, \bar{p}, q), (r, p, \bar{q}), (r, \bar{p}, \bar{q}), (\bar{r}, p, \bar{q}), (\bar{r}, \bar{p}, \bar{q})\}$	0.1

- **Marginalization** The knowledge, initially, encapsulated in the bba distribution can be refined as well as coarsened. The marginalization, which corresponds to a coarsening of knowledge, of an imperfect relation R_i defined on S_i, to a coarser set $S_i^{'}$,i.e.,$S_i \supseteq S_i^{'}$, is an imperfect relation $R_i^{(\downarrow S_i^{'})}$ defined on $S_i^{'}$ and obtained as follows:

$$m_i^{(\downarrow S_i^{'})}(\varphi) = \sum_{\theta \subseteq \Theta_i : \theta^{\downarrow S^{'}} i = \varphi} m_i(\theta) \text{ for all } \varphi \subseteq \Theta_i \quad (10)$$

Such that $\theta^{\downarrow S_i^{'}}$ denotes the projection of the set θ to $S_i^{'}$. We can employ the marginalization when we want to know to what extent a given partial instantiation, if extended, may satisfy the belief constraint C_i. For a closer examination, let us take the imperfect relation R_2 describing the belief constraint C_2 defined on $S_2 = \{Qu, Pa\}$ (see Table 4), its marginalization to $S_2^{'} = \{Qu\}$ gives the relation $R_2^{(\downarrow S_2^{'})}$ as follows (Table 4):

Table 4. The marginalized bba function for $R_2^{(\downarrow S_2^{'})}$

$2^{\Theta_2^{'}}$	$m_2^{(\downarrow S_2^{'})}$
$\{(q, \bar{q})\}$	$0.9 + 0.1 = 1$

3.3 Belief constraint

After expressing our beliefs on the imperfect relations in the credal level, we have to extract the satisfaction degree of the belief constraints by each instantiation θ in Θ_i aside using the pignistic probabilities produced by the TBM pignistic transformation of the mass distribution.
Let C_i be a belief constraint, R_i its relation and let m_i be the associated mass distribution over Θ_i, the produced pignistic probability, denoted by $BetP_i$, is defined as follows:

$$BetP_i(\theta) = \sum_{\varphi \subseteq \Theta_i} \left(m_i(\varphi) \frac{|\varphi \cap \theta|}{|\varphi|(1 - m_i(\emptyset))} \right) \text{,for all } \theta \in \Theta_i \quad (11)$$

Retaking our example "Pacifist or not", the computations are shown in the tables 5 and 6 for, respectively, C_1 and C_2.

Table 5. The satisfaction degrees of C_1

Θ_1	$BetP_1$
$\{(r, p)\}$	0.025
$\{(r, \bar{p})\}$	0.325
$\{(\bar{r}, p)\}$	0.325
$\{(\bar{r}, \bar{p})\}$	0.325

Hence, it can be easily seen that this notion of pignistic probability allows for expressing soft or flexible constraints starting from imperfect relations. It is of

Table 6. The satisfaction degrees of C_2

Θ_2	$BetP_2$
$\{(q,p)\}$	0.3325
$\{(q,\bar{p})\}$	0.0025
$\{(\bar{q},p)\}$	0.3325
$\{(\bar{q},\bar{p})\}$	0.3225

interest to discern the difference between hard constraint that should be certainly and fully satisfied and soft constraint whose satisfaction is not required to be neither certain nor total. Therefore, the satisfaction of a given constraint becomes, essentially, a matter of degree, such that:

$-BetP_i(\theta) = 1$ means that the instantiation θ totally satisfies the constraint C_i;
$-BetP_i(\theta) = 0$ means that the instantiation θ totally violates the constraint C_i;
$-0 < BetP_i(\theta) < 1$ means that the instantiation θ partially satisfies the constraint C_i;

The BetP also induces a preorder among instantiations. Formally, let θ and $\theta^{'}$ be two instantiations defined on the same set of variables, $BetP(\theta) > BetP(\theta^{'})$ means that θ is, a posteriori, preferred to $\theta^{'}$ for the satisfaction of the flexible belief constraint. Obviously, hard constraints are a particular case of belief constraints which are satisfied only to 1 or 0 degree. Hence the $BetP$ have also twofold role as it allows first expressing flexible constraints and second preferences among instantiations.

3.4 BCSP consistency

Belief constraint satisfiability. A belief constraint C_i, whose scope is S_i, is said to be (totally or partially) satisfied by a given instantiation $\theta \in \Theta_i$, noted $\theta \models C_i$ iff $BetP_i(\theta) > 0$. A belief constraint C_i is said to be unsatisfiable if there is no instantiation that satisfies it, i.e., $\forall\theta \in \Theta_i, BetP_i(\theta) = 0$.

Instantiation consistency. Classically, an instantiation θ of a set of variables $S \subseteq X$ is said to be consistent iff it satisfies all the constraints among that set. With the BCSP view, the constraint satisfiability is not any more a yes/no query but a matter of degree, where $BetP_i(\theta)$ indicates to what extent the instantiation θ satisfies the belief constraint C_i, and so the instantiation consistency is. Hence, a given instantiation θ is said to be consistent iff it satisfies all constraints among S to a non-nil degree. Formally, the consistency degree of an instantiation θ of a set of variables $S \subseteq X$ is obtained as follows[5]

$$C(\theta) = BetP_{\wedge}\{R_i|S_i \subseteq S\}(\theta) = \prod_{R_i^{\uparrow S}|S_i \subseteq S} \{BetP_i\}(\theta) = \prod_{R_i|S_i \subseteq S} \{BetP_i\}(\theta^{\downarrow S}) \quad (12)$$

− If θ totally satisfies all the constraints covering S, it said to be completely consistent, i.e.,$C(\theta) = 1$.
− If θ totally violates, at least, one constraint is said to be inconsistent, i.e., $C(\theta) = 0$.
− Otherwise, it is said to be partially consistent, i.e., $0 < C(\theta) < 1$.
As the BCSP is a generalization of the classical model, if the relations are perfect (total knowledge case), a given instantiation is either consistent (1) or inconsistent (0).

BCSP solution. A solution of a BCSP P (X, D, C, R) is every consistent complete instantiation θ, i.e., an instantiation of all the variables in X whose consistency degree is greater than 0, so that, all the constraints in C are satisfied. This consistency degree, evidently, corresponds to the satisfaction degree of the BCSP P by that instantiation.

$$S_P(\theta) = C(\theta) = \prod_{C_i \in C; R_i^{\uparrow X}} \{BetP_i\}(\theta) \quad (13)$$

Accordingly, we can merely notice that the satisfaction degree of the belief CSP, as defined above, accomplishes a sort of quantitative discrimination among the several instantiations inducing then a total preorder over them. Then, the higher is the satisfaction degree, the better is the instantiation. The solution space of a BCSP P (X, D, C, R) consists of the set of all the feasible solutions, i.e.,

$$Sols(P) = \{\theta \in D_1 \times ... \times D_n | S_P(\theta) > 0\} \quad (14)$$

BCSP consistency. A BCSP P (X, D, C, R) is said to be:
− Totally consistent if and only if it has at least one solution that totally satisfies all the constraints of C, i.e., $\exists \theta \in Sols(P) | S_P(\theta) = 1$.
− Totally inconsistent if and only if all instantiations of X are inconsistent, i.e., $Sols(P) = \emptyset$ or also $\forall \theta \in D_1 \times ... \times D_n | S_P(\theta) = 0$.
− Partially consistent if and only if all solutions are somehow feasible, i.e., $Sols(P) \neq \emptyset | \forall \theta \in Sols(P), S_P(\theta) < 1$.
Toward the same view, the consistency degree of a BCSP is the satisfaction degree of its best (optimal) solution, i.e.,

[5] The Fuzzy[2] and the Possibilistic[10] CSPs suffer from the "drowning effect" because of the egalitarian min-max operators use which barely discriminates between assignments that satisfy the CSP to the same degree. To avoid falling into the same weakness, we propose using an utilitarian operator, the product, to aggregate the preference degrees.

$$C(P) = S_P(\theta^*) = \max_{\theta \in Sols(P)} (S_P(\theta)) = \max_{\theta \in Sols(P)} \left(\prod_{C_i \in C; R_i^{\uparrow X}} \{BetP_i\}(\theta) \right) \quad (15)$$

As in the classical CSP, many tasks may be performed within the BCSP framework. The first and most intuitive task to perform may be to ascertain whether or not there is a solution, in other words find out if the problem is consistent or not. Given a satisfiable BCSP, some other tasks may be required such as: Determine the consistency degree of the problem, $C(P)$.
– Find any solution, i.e., some $\theta \in Sols(P) | S_P(\theta) > 0$.
– Find all of the solutions, $Sols(P)$.
– Find all solutions having a degree greater than a given bound B, i.e., every $\theta \in Sols(P) | S_P(\theta) > B$.
– Find an optimal solution, i.e., some $\theta \in Sols(P) | S_P(\theta) = C(P)$, and so forth.
Let us get back to our example "Pacifist or not". The local consistency of the instantiation $(Pa = \bar{p}, Re = r)$ is partial with respect to the constraint C_1, i.e., $C(Pa = \bar{p}, Re = r) = BetP_1(Pa = \bar{p}, Re = r) = 0.325$. The consistency of the complete instantiation $(Pa = p, Re = r, Qu = q)$ that indicates the satisfaction degree of P giving this is $C(Pa = p, Re = r, Qu = q) = S_P(Pa = p, Re = r, Qu = q) = \prod_{R_i^{\uparrow X} | S_i \subseteq X} \{BetP_i\}(Pa = p, Re = r, Qu = q) = 0.025 * 0.3325 = 0.008$. An optimal solution may be $(Pa = p, Re = \bar{r}, Qu = q)$, where, $C(Pa = p, Re = \bar{r}, Qu = q) = S_P(Pa = p, Re = \bar{r}, Qu = q) = 0.325 * 0.3325 = 0.108$. There are other possible solutions but less consistent such as $C(Pa = p, Re = r, Qu = \bar{q}) = 0.025 * 0.3325 = 0.008$ and $C(Pa = \bar{p}, Re = r, Qu = q) = 0.325 * 0.0025 = 0.0008$. The consistency of the problem is the consistency of its best solution, i.e., $C(P) = 0.108$. The problem is partially consistent.

4 Conclusion

In this paper, we have introduced the Belief Constraint Satisfaction Problem as a new CSP extension merging the CSP formalism and the belief function theory as interpreted by the TBM in order to deal with imperfect (uncertain and/or imprecise) relations and flexible constraints with the same formalism. This paper is, basically, devoted to represent the theoretical basis of the BCSP. By making the BCSP under experiments, we have proved some important results concerning the behavior of the BCSPs under uncertainty variation, besides, some interesting facts about links between constraints tightness [6] , constraints flexibility or softness and uncertainty. Furthermore, a comparison with the Transition Phase phenomenon[17] has been made. These evaluation results, besides, the comparison with other CSP extensions will be reported in a forthcoming paper. Further research targets exploiting the other informative measures offered by the belief functions theory in order to enlarge the scope of problems that can be tackled using the BCSP formalism such as the multi-objective optimization problems.

[6] The constraint tightness is the probability of an inconsistency between two values related by that constraint.

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